Short note on construction of gauge-invariant variables of linear metric perturbations on an arbitrary background spacetime

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Abstract

An outline of a proof of the decomposition of linear metric perturbations into gauge-invariant and gauge-variant parts on an arbitrary background spacetime which admits ADM decomposition is briefly discussed. We explicitly construct the gauge-invariant and gauge-variant parts of the linear metric perturbations based on some assumptions. This implies that we can develop the higher-order gauge-invariant perturbation theory on an arbitrary background spacetime.

Introduction — Higher-order general-relativistic perturbations have very wide applications. Among these applications, second-order cosmological perturbations are topical subject due to the precise measurements in recent cosmology. Higher-order black hole perturbations are also discussed in some literature. Moreover, as a special example of higher-order perturbation theory, there are researches on perturbations of a spherical star, which are motivated by researches into the oscillatory behaviors of a rotating neutron star. Thus, there are many physical situations to which general-relativistic higher-order perturbation theory should be applied.

As well-known, general relativity is based on the concept of general covariance. Due to this general covariance, the "gauge degree of freedom", which is unphysical degree of freedom of perturbations, arises in general-relativistic perturbations. To obtain physical results, we have to fix this gauge degrees of freedom or to extract some invariant quantities of perturbations. This situation becomes more complicated in higher-order perturbation theory. Therefore, it is worthwhile to investigate higher-order gauge-invariant perturbation theory from a general point of view.

According to these motivations, in Ref. [1], we proposed a procedure to find gauge-invariant variables for higher-order perturbations on an arbitrary background spacetime. This proposal is based on the single assumption that *we already know the procedure to find gauge-invariant variables for linear-order metric perturbations* (Conjecture 1 in this article). Under this assumption, we summarize some formulae for the second-order perturbations of the curvatures and energy-momentum tensor for the matter fields in Refs. [2, 3]. Confirming that the above assumption is correct in the case of cosmological perturbations, in Refs. [4], the second-order gauge-invariant cosmological perturbation theory was developed. Through these works, we find that our general framework of higher-order gauge-invariant perturbation theory is well-defined except for the above assumption for linear-order metric perturbations. Therefore, we proposed the above assumption as a conjecture in Ref. [3]. If this conjecture is true, our general-relativistic higher-order gauge-invariant perturbation theory is completely formulated on an arbitrary background spacetime and has very wide applications. The main purpose of this article is to give a brief outline of a proof of this conjecture. Details of this issue is given in Ref. [5].

Perturbations in general relativity — The notion of "gauge" in general relativity arise in the theory due to the general covariance. There are two kinds of "gauges" in general relativity. These two "gauges" are called as the first- and the second-kind gauges, respectively. The distinction of these two different notion of "gauges" is an important premise of our arguments. *The first-kind gauge* is a coordinate system on a single manifold $\mathcal{M}$. The coordinate transformation is also called *gauge transformation of the first kind* in general relativity. On the other hand, *the second-kind gauge* appears in perturbation theories in a theory with general covariance. In perturbation theories, we always treat two spacetime manifolds. One is the physical spacetime $\mathcal{M}$ which is our nature itself and we want to clarify the properties of

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$\mathcal{M}$ through perturbations. Another is the background spacetime $\mathcal{M}_0$ which has nothing to do with our nature but is prepared by hand for perturbative analyses. *The gauge choice of the second kind* is the point identification map $\mathcal{X} : \mathcal{M}_0 \mapsto \mathcal{M}$. We have to note that the correspondence $\mathcal{X}$ between points on $\mathcal{M}_0$ and $\mathcal{M}$ is not unique in the perturbation theory with general covariance. General covariance intuitively means that there is no preferred coordinate system in the theory. Due to this general covariance, we have no guiding principle to choose the identification map $\mathcal{X}$. Actually, as a gauge choice of the second kind, we may choose a different point identification map $\mathcal{Y}$ from $\mathcal{X}$. This implies that there is degree of freedom in the gauge choice of the second kind. This is *the gauge degree of freedom of the second kind* in perturbation theory. *The gauge transformation of the second kind* is understood as a change $\mathcal{X} \to \mathcal{Y}$ of the identification map.

To define perturbations of an arbitrary tensor field $\bar{Q}$, we have to compare $\bar{Q}$ on the physical spacetime $\mathcal{M}_\lambda$ with Q_0 on the background spacetime $\mathcal{M}_0$ through the introduction of the above second-kind gauge choice $\mathcal{X}_\lambda : \mathcal{M}_0 \to \mathcal{M}_\lambda$. The pull-back $\mathcal{X}^*_\lambda$, which is induced by the map $\mathcal{X}_\lambda$, maps a tensor field $\bar{Q}$ on $\mathcal{M}_\lambda$ to a tensor field $\mathcal{X}^*_\lambda \bar{Q}$ on $\mathcal{M}_0$. Once the definition of the pull-back of the gauge choice $\mathcal{X}_\lambda$ is given, the perturbations of a tensor field $\bar{Q}$ under the gauge choice $\mathcal{X}_\lambda$ are simply defined by the evaluation of the Taylor expansion at $\mathcal{M}_0$:

$$ {}^{\mathcal{X}}Q := \mathcal{X}^*_\lambda \bar{Q}_\lambda\big|_{\mathcal{M}_0} = Q_0 + \lambda {}^{(1)}_{\mathcal{X}}Q + \frac{1}{2}\lambda^2 {}^{(2)}_{\mathcal{X}}Q + O(\lambda^3), \tag{1} $$

where ${}^{(1)}_{\mathcal{X}}Q$ and ${}^{(2)}_{\mathcal{X}}Q$ are the first- and the second-order perturbations of $\bar{Q}$, respectively.

When we have two different gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$, we have two different representations of the perturbative expansion of the pulled-backed variables $\mathcal{X}^*_\lambda \bar{Q}_\lambda\big|_{\mathcal{M}_0}$ and $\mathcal{Y}^*_\lambda \bar{Q}_\lambda\big|_{\mathcal{M}_0}$. Although these two representations of the perturbations are different from each other, these should be equivalent because of general covariance. This equivalence is guaranteed by the *gauge-transformation rules* between two different gauge choices. The change of the gauge choice from $\mathcal{X}_\lambda$ to $\mathcal{Y}_\lambda$ is represented by the diffeomorphism $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda$. This diffeomorphism Φ_λ is the map $\Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$ for each value of $\lambda \in \mathbb{R}$ and does change the point identification. Therefore, the diffeomorphism Φ_λ is regarded as the gauge transformation $\Phi_\lambda : \mathcal{X}_\lambda \to \mathcal{Y}_\lambda$. The gauge transformation Φ_λ induces a pull-back from the representation ${}^{\mathcal{X}}Q_\lambda$ of the perturbed tensor field Q in the gauge choice $\mathcal{X}_\lambda$ to the representation ${}^{\mathcal{Y}}Q_\lambda$ in the gauge choice $\mathcal{Y}_\lambda$. Actually, the tensor fields ${}^{\mathcal{X}}Q_\lambda$ and ${}^{\mathcal{Y}}Q_\lambda$, which are defined on $\mathcal{M}_0$, are connected by the linear map Φ^*_λ as ${}^{\mathcal{Y}}Q_\lambda = \Phi^*_\lambda {}^{\mathcal{X}}Q_\lambda$. According to generic arguments concerning the Taylor expansion of the pull-back of tensor fields on the same manifold [6], we obtain the order-by-order gauge-transformation rules for the perturbative variables ${}^{(1)}Q$ and ${}^{(2)}Q$ as

$$ {}^{(1)}_{\mathcal{Y}}Q - {}^{(1)}_{\mathcal{X}}Q = \pounds_{\xi_{(1)}} Q_0, \quad {}^{(2)}_{\mathcal{Y}}Q - {}^{(2)}_{\mathcal{X}}Q = 2\pounds_{\xi_{(1)}} {}^{(1)}_{\mathcal{X}}Q + \left\{ \pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}} \right\} Q_0. \tag{2} $$

where the vector fields $\xi^a_{(1)}$ and $\xi^a_{(2)}$ are the generators of the gauge transformation Φ_λ.

The notion of gauge invariance considered in this article is the *order-by-order gauge invariance* proposed in Ref. [3]. We call the kth-order perturbation ${}^{(k)}_{\mathcal{X}}Q$ is gauge invariant iff ${}^{(k)}_{\mathcal{X}}Q = {}^{(k)}_{\mathcal{Y}}Q$ for any gauge choice $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$. Through this concept of the order-by-order gauge invariance, we can develop the gauge-invariant perturbation theory.

Construction of gauge-invariant variables — First, we consider the metric perturbation. The metric $\bar{g}_{ab}$ on $\mathcal{M}$, which is pulled back to $\mathcal{M}_0$ using a gauge choice $\mathcal{X}_\lambda$, is expanded in the form of Eq. (1): $\mathcal{X}^*_\lambda \bar{g}_{ab} = g_{ab} + \lambda {}_{\mathcal{X}}h_{ab} + (\lambda^2/2) {}_{\mathcal{X}}l_{ab} + O^3(\lambda)$, where g_{ab} is the metric on $\mathcal{M}_0$. Although this expansion of the metric depends entirely on the gauge choice $\mathcal{X}_\lambda$, henceforth, we do not explicitly express the index of the gauge choice $\mathcal{X}_\lambda$ if there is no possibility of confusion. Through these setup, in Ref. [1], we proposed a procedure to construct gauge-invariant variables for higher-order perturbations. Our starting point to construct gauge-invariant variables is the following conjecture for the linear-order metric perturbation h_{ab} defined by the above metric expansion:

Conjecture 1. *If there is a symmetric tensor field h_{ab} of the second rank, whose gauge transformation rule is ${}_{\mathcal{Y}}h_{ab} - {}_{\mathcal{X}}h_{ab} = \pounds_{\xi_{(1)}} g_{ab}$, then there exist a tensor field $\mathcal{H}_{ab}$ and a vector field X^a such that h_{ab} is decomposed as $h_{ab} =: \mathcal{H}_{ab} + \pounds_X g_{ab}$, where $\mathcal{H}_{ab}$ and X^a are transformed as ${}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0$, ${}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi^a_{(1)}$ under the gauge transformation (2), respectively.*

In this conjecture, $\mathcal{H}_{ab}$ and X^a are *gauge-invariant* and *gauge-variant* parts of the perturbation h_{ab}. In the case of the perturbation theory on an arbitrary background spacetime, this conjecture is a highly non-trivial statement due to the non-trivial curvature of the background spacetime, though its inverse statement is trivial.

An outline of a proof of Conjecture 1 — To give an outline of a proof of Conjecture 1 on an arbitrary background spacetime, we assume that the background spacetimes admit ADM decomposition. Therefore, the background spacetime $\mathcal{M}_0$ considered here is $n+1$-dimensional spacetime which is described by the direct product $\mathbb{R}\times\Sigma$. Here, $\mathbb{R}$ is a time direction and Σ is the spacelike hypersurface ($\dim\Sigma = n$) embedded in $\mathcal{M}_0$. This means that $\mathcal{M}_0$ is foliated by the one-parameter family of spacelike hypersurface $\Sigma(t)$, where $t\in\mathbb{R}F$ is a time function. Then, the metric on $\mathcal{M}_0$ is described by

$$g_{ab} \quad = \quad -\alpha^2(dt)_a(dt)_b + q_{ij}(dx^i+\beta^i dt)_a(dx^j+\beta^j dt)_b, \tag{3}$$

where α is the lapse function, β^i is the shift vector, and $q_{ab} = q_{ij}(dx^i)_a(dx^j)_b$ is the metric on $\Sigma(t)$.

Since the ADM decomposition (3) is a local one, we may regard that the arguments in this article are restricted to that for a single patch in $\mathcal{M}_0$ which is covered by the metric (3). Further, we may change the region which is covered by the metric (3) through the choice of the lapse function α and the shift vector β^i. The choice of α and β^i is regarded as the first-kind gauge choice, which have nothing to do with the second-kind gauge. Since we may regard that the representation (3) of the background metric is that on a single patch in $\mathcal{M}_0$, in general situation, each Σ may have its boundary $\partial\Sigma$.

To prove Conjecture 1, we first consider the components of the metric h_{ab} as $h_{ab} = h_{tt}(dt)_a(dt)_b + 2h_{ti}(dt)_{(a}(dx^i)_{b)} + h_{ij}(dx^i)_a(dx^j)_b$. Under the gauge-transformation rule ${}_{\mathcal{Y}}h_{ab} - {}_{\mathcal{X}}h_{ab} = \pounds_{\xi_{(1)}}g_{ab}$, the components $\{h_{tt},\, h_{ti},\, h_{ij}\}$ are transformed as

$$\begin{aligned}
{}_{\mathcal{Y}}h_{tt} - {}_{\mathcal{X}}h_{tt} \quad &= \quad 2\partial_t\xi_t - \frac{2}{\alpha}\left(\partial_t\alpha + \beta^i D_i\alpha - \beta^j\beta^i K_{ij}\right)\xi_t \\
&\quad -\frac{2}{\alpha}\left(\beta^i\beta^k\beta^j K_{kj} - \beta^i\partial_t\alpha + \alpha q^{ij}\partial_t\beta_j + \alpha^2 D^i\alpha - \alpha\beta^k D^i\beta_k - \beta^i\beta^j D_j\alpha\right)\xi_i, \qquad (4)\\
{}_{\mathcal{Y}}h_{ti} - {}_{\mathcal{X}}h_{ti} \quad &= \quad \partial_t\xi_i + D_i\xi_t - \frac{2}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right)\xi_t - \frac{2}{\alpha}M_i{}^j\xi_j, \qquad (5)\\
{}_{\mathcal{Y}}h_{ij} - {}_{\mathcal{X}}h_{ij} \quad &= \quad 2D_{(i}\xi_{j)} + \frac{2}{\alpha}K_{ij}\xi_t - \frac{2}{\alpha}\beta^k K_{ij}\xi_k, \qquad (6)
\end{aligned}$$

where $M_i{}^j$ is defined by $M_i{}^j := -\alpha^2 K^j{}_i + \beta^j\beta^k K_{ki} - \beta^j D_i\alpha + \alpha D_i\beta^j$. Here, K_{ij} is the components of the extrinsic curvature of Σ in $\mathcal{M}_0$ and D_i is the covariant derivative associate with the metric q_{ij} ($D_i q_{jk} = 0$). The extrinsic curvature K_{ij} and its trace K are related to the time derivative of the metric q_{ij} by $K_{ij} = -(1/2\alpha)\left[\partial_t q_{ij} - D_i\beta_j - D_j\beta_i\right]$ and $K := q^{ij}K_{ij}$, respectively.

Inspecting gauge-transformation rules (4)–(6), we consider the decomposition of the components $\{h_{ti}, h_j\}$ into the set of the variables $\{h_{(VL)},\, h_{(V)i},\, h_{(L)},\, h_{(TV)i},\, h_{(TT)ij}\}$ as follows:

$$\begin{aligned}
&h_{ti} =: D_i h_{(VL)} + h_{(V)i} - \frac{2}{\alpha}\left(D_i\alpha - \beta^k K_{ik}\right)\left(h_{(VL)} - \Delta^{-1}D^k\partial_t h_{(TV)k}\right) - \frac{2}{\alpha}M_i{}^k h_{(TV)k}, \qquad (7)\\
&h_{ij} =: \frac{1}{n}q_{ij}h_{(L)} + D_i h_{(TV)j} + D_j h_{(TV)i} - \frac{2}{n}q_{ij}D^k h_{(TV)k} + h_{(TT)ij} \\
&\qquad + \frac{2}{\alpha}K_{ij}\left(h_{(VL)} - \Delta^{-1}D^k\partial_t h_{(TV)k}\right) - \frac{2}{\alpha}K_{ij}\beta^k h_{(TV)k}, \qquad (8)\\
&D^i h_{(V)i} = 0, \quad q^{ij}h_{(TT)ij} = 0 = D^i h_{(TT)ij}. \qquad (9)
\end{aligned}$$

Here, we assume the existence of Green functions of the elliptic derivative operators $\Delta := D^i D_i$ and $\mathcal{F} := \Delta - \frac{2}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right)D^i - 2D^i\left\{\frac{1}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right)\right\}$, and the existence and the uniqueness of the solution A_i to the integro-differential equation

$$\mathcal{D}_j{}^k A_k + D^m\left[\frac{2}{\alpha}\tilde{K}_{mj}\left\{\mathcal{F}^{-1}D^k\left(\frac{2}{\alpha}M_k{}^l A_l - \partial_t A_k\right) - \beta^k A_k\right\}\right] = L_j \tag{10}$$

for given a vector field L_j. Although the derivation of Eq. (7)–(9) is highly non-trivial, we only note that the relation (7)–(9) between the variables $\{h_{ti}, h_{ij}\}$ and $\{h_{(VL)},\, h_{(V)i},\, h_{(L)},\, h_{(TV)i},\, h_{(TT)ij}\}$ is invertible

if we accept above three assumptions. In other words, the fact that we based on these assumptions implies that we have ignored perturbative modes which belong to the kernel of the above derivative operators and trivial solutions to Eq. (10) if there exists. These modes should be separately treated in different manner. We call these modes as *zero modes*. The issue concerning about treatments of these zero modes is called *zero-mode problem*, which is a remaining problem in our general framework on higher-order general-relativistic gauge-invariant perturbation theory.

Under these assumptions, the gauge-transformation rules for the variables $\{h_{(VL)}, h_{(V)i}, h_{(L)}, h_{(TV)i}, h_{(TT)ij}\}$ are summarized as follows:

$$
{}_{\mathcal{Y}}h_{(VL)} - {}_{\mathcal{X}}h_{(VL)} = \xi_t + \Delta^{-1}D^k\partial_t\xi_k, \quad {}_{\mathcal{Y}}h_{(V)i} - {}_{\mathcal{X}}h_{(V)i} = \partial_t\xi_i - D_i\Delta^{-1}D^k\partial_t\xi_k, \tag{11}
$$

$$
{}_{\mathcal{Y}}h_{(L)} - {}_{\mathcal{X}}h_{(L)} = 2D^i\xi_i, \quad {}_{\mathcal{Y}}h_{(TV)l} - {}_{\mathcal{X}}h_{(TV)l} = \xi_l, \quad {}_{\mathcal{Y}}h_{(TT)ij} - {}_{\mathcal{X}}h_{(TT)ij} = 0. \tag{12}
$$

From these gauge-transformation rules, we may define the components of the gauge-variant part X_a by $X_i := h_{(TV)i}$ and $X_t := h_{(VL)} - \Delta^{-1}D^k\partial_t h_{(TV)k}$. Then, we obtain the gauge-variant part X_a of the perturbation h_{ab} as $X_a := X_t(dt)_a + X_i(dx^i)_a$. Using the above variables X_t and X_i, we can construct gauge-invariant variables for the linear-order metric perturbation h_{ab}:

$$
\begin{aligned}
-2\Phi \;&:=\; h_{tt} + \frac{2}{\alpha}\left(\partial_t\alpha + \beta^i D_i\alpha - \beta^j\beta^i K_{ij}\right)X_t - 2\partial_t X_t \\
&\quad + \frac{2}{\alpha}\left(\beta^i\beta^k\beta^j K_{kj} - \beta^i\partial_t\alpha + \alpha q^{ij}\partial_t\beta_j + \alpha^2 D^i\alpha - \alpha\beta^k D^i\beta_k - \beta^i\beta^j D_j\alpha\right)X_i,
\end{aligned}\tag{13}
$$

$$
-2n\Psi \;:=\; h_{(L)} - 2D^iX_i, \quad \nu_i := h_{(V)i} - \partial_t X_i + D_i\Delta^{-1}D^k\partial_t X_k, \quad \chi_{ij} := h_{(TT)ij}. \tag{14}
$$

Actually, we can easily confirm that these variables Φ, Ψ, ν_i, and χ_{ij} are gauge invariant. We also note that the variable ν_i satisfies the property $D^i\nu_i = 0$ and the variable χ_{ij} satisfies the properties $\chi_{ij} = \chi_{ji}$, $q^{ij}\chi_{ij} = 0$, and $D^i\chi_{ij} = 0$. The original components $\{h_{tt}, h_{ti}, h_{ij}\}$ of the metric perturbation h_{ab} is rewritten in terms of these gauge-invariant variables and the variables X_t and X_i. These representation shows that we may define the gauge-invariant variables $\mathcal{H}_{ab}$ so that $\mathcal{H}_{ab} := -2\Phi(dt)_a(dt)_b + 2\nu_i(dt)_{(a}(dx^i)_{b)} + \left(-2\Psi q_{ij} + \chi_{ij}\right)(dx^i)_{(a}(dx^j)_{b)}$. This leads to assertion of Conjecture 1. □

Summary — We proposed an outline of a proof of Conjecture 1 for an arbitrary background spacetime. Conjecture 1 states that we already know the procedure to decompose the linear-order metric perturbation h_{ab} into its gauge-invariant part $\mathcal{H}_{ab}$ and gauge-variant part X_a. Conjecture 1 is the only non-trivial part when we consider the general framework of gauge-invariant perturbation theory on an arbitrary background spacetime. Although there will be many approaches to prove Conjecture 1, in this article, we just proposed an outline a proof. We also note that our arguments do not include zero modes. The existence of zero modes is also related to the symmetry of the background spacetime. To resolve this zero-mode problem, careful discussions on domains of functions for perturbations and its boundary conditions at $\partial\Sigma$ will be necessary. Besides this zero-mode problem, we have almost completed the general framework of the general-relativistic higher-order gauge-invariant perturbation theory. The outline of a proof of Conjecture 1 shown in this article gives rise to the possibility of the application of our general framework not only to cosmological perturbations [3, 4] but also to perturbations of black hole spacetimes or perturbations of general relativistic stars. Therefore, we may say that the wide applications of our gauge-invariant perturbation theory will be opened. We leave these development as future works.

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